METHOD FOR CALCULATING THE VELOCITY FIELD AND THE "SUSPENSION" MODEL OF A NONLINEARLY VISCOUS LIQUID IN CYLINDRICAL CHANNELS WITH AN ARBITRARY TRANSVERSE CROSS SECTION

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A method is proposed for calculating the velocity field of a nonlinearly viscous liquid in a cylindrical channel with an arbitrary transverse cross section. It is proved that a suspension model exists for such a liquid under the conditions of the interior problem.

Formulation of the Problem. In the hydromechanics of a nonlinearly viscous liquid there exists a well known and interesting problem of great practical value: the problem of the velocity distribution in a laminar steady-state flow in a cylindrical channel with an arbitrary transverse cross section.

The system of equations of motion and continuity, describing this problem, can be written as

$$\frac{\partial}{\partial \chi_1}\left(\mu(I_2)\frac{\partial V}{\partial \chi_1}\right)+\frac{\partial}{\partial \chi_2}\left(\mu(I_2)\frac{\partial V}{\partial \chi_2}\right)=-\frac{\partial P}{\partial z}=\mathrm{const} \tag{1}$$

with the boundary condition

$$V(\chi_\alpha)|_\Gamma=0,\quad \alpha=1,\ 2, \tag{2}$$

where the second invariant of the strain-rate tensor has the form

$$I_2=\left(\frac{\partial V}{\partial \chi_1}\right)^2+\left(\frac{\partial V}{\partial \chi_2}\right)^2. \tag{3}$$

It is well known that an explicit solution of the problem (1)-(3) with arbitrary $\mu(I_2)$ does not exist. It is also well known [1] that the solution of the problem posed, consisting of finding the actual velocity field based on the principle of least action, is equivalent to finding the minimum of the functional

$$F(V)=\iint\limits_{\Omega} d\chi_1 d\chi_2\int\limits_0^{I_2}\mu(\xi)\,d\xi+2\frac{\partial P}{\partial z}\iint\limits_{\Omega} V d\chi_1 d\chi_2. \tag{4}$$

The functions which furnish the functional F(V) a minimum are usually found numerically by variational methods [2]. The variational approach to the problem (1)-(3), however, has a number of drawbacks. These are, first of all, the complexity of the choice of basis functions for regions with partial symmetry or no symmetry at all and, second, the quite cumbersome algorithm for numerical implementation. For asymmetric regions, as a rule, it is possible to construct several variants of basis functions with further testing by numerical means. In addition, the lack of symmetry increases the number of equations in the nonlinear system (4) and, consequently, it increases the computational time [3].

We propose below a method for solving the problem (1)-(3) without the indicated drawbacks and a much better, than the variational approach, rate of convergence.

Method of Solution and Its Justification. In a previous publication [4] it was proposed that the problem (1)-(3) be solved by replacing the components of the shear stress by the expressions

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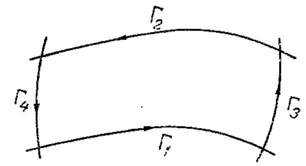

Fig. 1

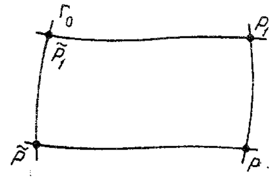

Fig. 2

Fig. 1. Region of interest Ω with the boundary Γ. Reduction to the case of the four-corner contour.

Fig. 2. Orthogonal coordinate system in the region of interest.

$$\mu(I_2)\frac{\partial V}{\partial \chi_1}=\frac{1}{2}\frac{\partial P}{\partial z}\frac{\partial U}{\partial \chi_1},\quad \mu(I_2)\frac{\partial V}{\partial \chi_2}=\frac{1}{2}\frac{\partial P}{\partial z}\frac{\partial U}{\partial \chi_2}. \tag{5}$$

After the substitution (5) the starting nonlinear equation (1) is linearized into the Poisson equation, which is correct only for regions with full symmetry – rings and strips. In other cases, however, this approach will describe a Newtonian velocity distribution [5].

In order that the substitution (5) approach as close as possible the case of a flow of a nonlinearly viscous liquid of interest here and in order to obtain an explicit solution in the ideal case, we shall study the following substitution:

$$\mu(I_2)\frac{\partial V}{\partial \chi_1}=\lambda(\chi_\alpha)\frac{\partial U}{\partial \chi_1},\quad \mu(I_2)\frac{\partial V}{\partial \chi_2}=\lambda(\chi_\alpha)\frac{\partial U}{\partial \chi_2}. \tag{6}$$

Here the term $(1/2)(\partial P/\partial z)$ is omitted in order to simplify the notation.

We assume that $\lambda(\chi_\alpha) > 0$ (otherwise, as a result of the substitution (6), we shall obtain from (1) a degenerate elliptic equation). We assume also [3] that $\mu(I_2)$ depends monotonically on I_2. In order for the substitution (6) to be valid, it is necessary (and in a simply connected region sufficient also) that

$$\frac{\partial}{\partial \chi_1}\left(\frac{1}{\lambda(\chi_\alpha)}\mu(I_2)\frac{\partial V}{\partial \chi_2}\right)=\frac{\partial}{\partial \chi_2}\left(\frac{1}{\lambda(\chi_\alpha)}\mu(I_2)\frac{\partial V}{\partial \chi_1}\right). \tag{7}$$

We denote

$$\ln\mu(I_2)=M(I_2)\quad\text{or}\quad \mu(I_2)=e^{M(I_2)},\qquad \ln\lambda(\chi_\alpha)=\Lambda(\chi_\alpha)\quad\text{or}\quad \lambda(\chi_\alpha)=e^{\Lambda(\chi_\alpha)}.$$

This can be done, since $\mu > 0$ and μ is a monotonic function, while $\lambda > 0$. Then from (7)

$$-e^{-\Lambda}\Lambda_{\chi_2}e^{M}\frac{\partial V}{\partial \chi_1}+e^{-\Lambda}M_{\chi_2}e^{M}\frac{\partial V}{\partial \chi_1}+e^{-\Lambda}e^{M}\frac{\partial^2 V}{\partial \chi_1\partial \chi_2}=-e^{-\Lambda}\Lambda_{\chi_1}e^{M}\frac{\partial V}{\partial \chi_2}+e^{-\Lambda}M_{\chi_1}e^{M}\frac{\partial V}{\partial \chi_2}+e^{-\Lambda}e^{M}\frac{\partial^2 V}{\partial \chi_1\partial \chi_2}.$$

We denote $\partial V/\partial \chi_1 = p$, $\partial V/\partial \chi_2 = q$ and making the cancellations, we obtain

$$M_{\chi_1}q-M_{\chi_2}p=\Lambda_{\chi_1}q-\Lambda_{\chi_2}p. \tag{8}$$

The vector $(q, -p)$ is perpendicular to (p, q) (indeed, $qp + (-pq) = 0$) and therefore to $\left(\frac{\partial U}{\partial \chi_1},\frac{\partial U}{\partial \chi_2}\right)$ because of their proportionality. Therefore, the vector $(q, -p)$ is oriented toward the tangent to the conour line of the function $V(\chi_\alpha)$ (and $U(\chi_\alpha)$), passing through the point (χ_α). The equality (8) means that the derivatives of M and Λ in the direction of the contour line are equal. From here it follows that

$$M(I_2(\chi_\alpha))=\Lambda(\chi_\alpha)+\ln C(\chi_\alpha),$$

where $C(\chi_\alpha)$ is constant on the contour lines of the function $V(\chi_\alpha)(U(\chi_\alpha))$. Then, after taking the algorithm

$$\mu(I_2(\chi_\alpha))=\lambda(\chi_\alpha)C(\chi_\alpha). \tag{9}$$

It follows from (9) that for any region Ω it is possible to find $\lambda(\chi_\alpha)$ such that the substitutions (6) for Eq. (1) are valid, i.e.,

$$\lambda(\chi_\alpha)=\frac{1}{C(\chi_\alpha)}\mu\left(\left(\frac{\partial V}{\partial \chi_1}\right)^2+\left(\frac{\partial V}{\partial \chi_2}\right)^2\right).$$

Thus, it has been shown here that the function $\lambda(\chi_\alpha)$ existing for any simply connected region Ω depends on the form of the given region (dependence on $C(\chi_\alpha)$) and on the specific form of the rheological model of the liquid (dependence on $\mu(I_2)$).

The problem is to find the form of the function $\lambda(\chi_\alpha)$. Since the problem of the precise choice of $\lambda(\chi_\alpha)$ for a known region is no simpler than the integration of the starting nonlinear equation, we shall study the inverse problem or those regions in which the substitution (6) is justified with a fixed function $\lambda(\chi_\alpha)$.

It follows from (9) that $I_2(\chi_\alpha) = \mu_{-1}(\lambda(\chi_\alpha)C(\chi_\alpha))$, where μ_{-1} is the function inverse to μ (μ is by definition monotonic).

Since according to (6) the gradients of $V(\chi_\alpha)$ and $U(\chi_\alpha)$ are proportional, the contour lines of one of these functions are automatically contour lines of the other also. Since $\lambda(\chi_\alpha)$ is fixed, $U(\chi_\alpha)$ will be the solution of the well-known problem

$$\frac{\partial}{\partial\chi_1}\left(\lambda(\chi_\alpha)\frac{\partial U}{\partial\chi_1}\right)+\frac{\partial}{\partial\chi_2}\left(\lambda(\chi_\alpha)\frac{\partial U}{\partial\chi_2}\right)=\text{const}, \qquad U|_\Gamma=0, \tag{10}$$

and it may therefore be assumed that the contour lines of $U(\chi_\alpha)$ have been found.

The condition that (p, q) be the gradient of some function (which will automatically be the function $V(\chi_\alpha)$ — this follows from the uniqueness of the solution of (1)-(3)) consists of the condition that

$$\oint_\Gamma p(\chi_\alpha)\,d\chi_1+q(\chi_\alpha)\,d\chi_2=0 \tag{11}$$

must be potential for any closed contour Γ in Ω. As usual, Γ can be regarded as being "broken," and consisting of pieces of contour lines of $U(\chi_\alpha)$ and pieces of trajectories orthogonal to them. The problem reduces to the case of the four-corner contour shown in Fig. 1.

Since the vector (p, q) is perpendicular to the contour lines, the integrals along Γ_1 and Γ_2, which can be interpreted as the work, equal zero. For this reason the integrals along Γ_3 and Γ_4 (i.e., along trajectories orthogonal to the contour lines of $U(\chi_\alpha)$) equal

$$\int_{\overrightarrow{\Gamma_3}} p\,d\chi_1+q\,d\chi_2=\int_{\Gamma_3}\sqrt{p^2+q^2}\,dS,$$

$$\int_{\overrightarrow{\Gamma_4}} p\,d\chi_1+q\,d\chi_2=\int_{\Gamma_4}\sqrt{p^2+q^2}\,dS.$$

From here, we can write the following in terms of the function inverse to μ:

$$\int_{\Gamma_3}\sqrt{\mu_{-1}(\lambda(\chi_\alpha)C(\chi_\alpha))}\,dS=\int_{\Gamma_4}\sqrt{\mu_{-1}(\lambda(\chi_\alpha)C(\chi_\alpha))}\,dS, \tag{12}$$

i.e., the condition that (11) must be potential has transformed into (12).

We shall use the orthogonal coordinate system introduced above — $U(\chi_\alpha)$ and the trajectories orthogonal to them. Let P be a point in Ω, Γ_0 is the line from a second family of lines of the coordinate system, $\tilde{P}$ is a point lying on Γ_0 and on the same contour line as P (Fig. 2).

Since the integral equality (12) can be checked on small integration segments, it may be assumed that the integrand varies insignificantly on this segment. Therefore the integral is the product of the value of the function and the length of the integration interval. Transferring these lengths of two segments or orthogonal trajectories confined between close contour lines to one side of the equality, we obtain

$$\frac{\overset{\smile}{PP_1}}{\overset{\smile}{\tilde{P}\tilde{P}_1}}=\sqrt{\frac{\mu_{-1}(\lambda(\tilde{P})C(\tilde{P}))}{\mu_{-1}(\lambda(P)C(P))}}.$$

Then the quantity

$$\dot{\gamma}(P,\,\Gamma_0)=\lim_{P_1\to P}\frac{\overset{\smile}{PP_1}}{\overset{\smile}{\tilde{P}\tilde{P}_1}}$$

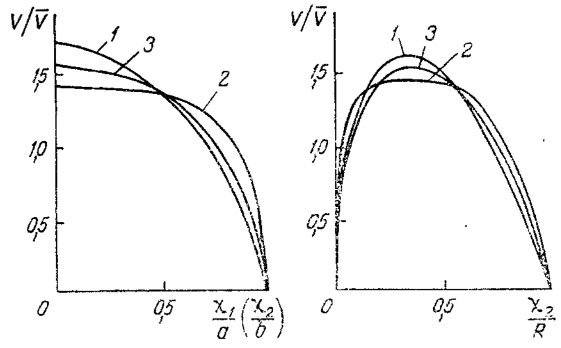

Fig. 3. Computed dimensionless velocity profiles along the axes of symmetry of the rectangular and semicircular channels: 1) Newtonian profile; 2) non-Newtonian profile; 3) velocity profile at the first iteration after introducing the function $\lambda(\chi_\alpha)$.

can be called the divergence (fan) of the contour lines at P referred to Γ_0. Here P_1 lies on the line from the second family of coordinate lines, passing through P; P_1 is the corresponding point on Γ_0.

Thus, the condition (12) is equivalent to the equality

$$\mu_{-1}(\lambda(\tilde{P})C(\tilde{P})) = \gamma(P,\ \Gamma_0)\,\mu_{-1}(\lambda(P)C(P)), \tag{13}$$

i.e., for every contour line there exists a number C such that $\gamma(P,\ \Gamma_0)\mu_{-1}(\lambda(P)C)$ is independent of the position of P on the contour line. This is the necessary and sufficient condition for the applicability of the substitution (6) with the given function $\lambda(\chi_\alpha)$ to the problem (1)-(3) in the region Ω.

The above analysis showed that for any region Ω there always exists a function $\lambda(\chi_\alpha)$ which enables the explicit solution of the problem formulated with the help of the substitution (6). It was shown above that the form of $\lambda(\chi_\alpha)$ depends on the form of the region and on the rheological model of the non-Newtonian liquid. If $\lambda(\chi_\alpha)$ can be found, then the nonlinear, relative to $V(\chi_\alpha)$ equation reduces to a linear equation for $U(\chi_\alpha)$ and to the algebraic system (6).

As already mentioned, the problem of finding an exact expression for $\lambda(\chi_\alpha)$ is no simpler than the problem of integrating the starting nonlinear equation, but an approximate choice of $\lambda(\chi_\alpha)$, based only on the geometry of the region, already makes it possible to improve substantially the first iteration in the numerical realization of the problem and substantially shorten the calculations of the velocity field.

We shall illustrate this with a numerical example. Consider the flow of a model Newtonian liquid (the solution Na–carboxymethyl nitrocellulose) in a rectangular channel with sides $2a = 0.18$ m and $2b = 0.015$ m and a semicircular channel with radius $R = 0.006$ m with $\partial P/\partial z = 600\ \mathrm{H/m^3}$. For the function $\mu(I_2)$ we shall use the generalized rheological Kutateladze–Khabakhpasheva law [6] for a structurally viscous non-Newtonian liquid $d\varphi_* = -\varphi_*^n d\tau_*$ in the particular, or greatest practical interest form

$$\varphi_* = \exp(-\tau_*), \tag{14}$$

where $\varphi_* = (\varphi_\infty - \varphi)/(\varphi_\infty - \varphi_0)$; $\tau_* = \theta(\tau - \tau_1)/(\varphi_\infty - \varphi_0)$.

The parameters of the model liquid are as follows: $\theta = 0.1981(\mathrm{Pa^2\cdot sec})^{-1}$; $\varphi_0 = 1.9\ (\mathrm{Pa\cdot sec})^{-1}$; $\varphi_\infty = 13.7\ (\mathrm{Pa\cdot sec})^{-1}$; $\tau_1 = 0$.

The dimensions of the channels and the conditions of flow are identical to those given in [3]. The functions $U(\chi_\alpha)$ for both channels are also given in [3].

The functions $\lambda(\chi_\alpha)$, based on the geometry of the region, are chosen in the following form: for a rectangle

$$\lambda(\chi_\alpha) = \frac{1}{a^2}\left(\chi_1 - \frac{a}{\chi_1}\right)^2 + \frac{1}{b^2}\left(\chi_2 - \frac{b}{\chi_2}\right)^2, \tag{15}$$

for a semicircle

$$\lambda(\chi_\alpha) = \left(\frac{r}{R}\right)^4. \tag{16}$$

The algorithm of the numerical realization of the problem with the substitution (6) is completely analogous and is described in detail in [3], where the substitution (5) is used for the solution. The difference lies only in the calculation of the matrix $\lambda(\chi_\alpha)$ (simultaneously with $\partial U/\partial\chi_\alpha$).

As the numerical calculations showed, in order to achieve a relative error of $\varepsilon = 10^{-4}$ in the calculation of the flow in a rectangular channel three exterior iterations in the non-Newtonian viscosity and three internal iterations are required for finding $V(\chi_\alpha)$ by the method of variable directions (MVD).

The following results were obtained for a semicircular channel three external and four internal iterations.

The somewhat slower rate of convergence for the semicircular channel is a result of the fact that there is no symmetry with respect to one of the axes and the corresponding behavior of $U(\chi_\alpha)$.

Curves of the velocities for both types of channels were constructed from the computational results (Fig. 3). The curves 1 in Fig. 3 correspond to the Newtonian velocity profile, obtained with the help of the substitution (5) at the first step of iteration. The curves 2 correspond to the non-Newtonian velocity profile with $\varepsilon = 10^{-4}$. The curves 3 correspond to the velocity profile obtained after the first iteration with the help of the substitution (6). As is evident from Fig. 3, the curves 3 correspond much more closely than the Newtonian profile to the non-Newtonian velocity distribution sought.

Thus, the proposed method for calculating the velocity field in cylindrical channels with an arbitrary transverse cross section introducing the function $\lambda(\chi_\alpha)$ is distinguished by a high rate of convergence, short computing time (which is important in solving systems of equations of heat and mass transport), and the simplicity of the algorithm.

In the future, tables of approximate functions $\lambda(\chi_\alpha)$ will be constructed for a number of regions and rheological laws most often encountered in practice, and a method for finding $\lambda(\chi_\alpha)$ will be presented.

"Suspension" Model of a Nonlinearly Viscous Liquid. Aside from the improvement of the iteration approach to the solution of hydrodynamic nonlinear internal problems, the method developed here enables the representation of the internal flow of a nonlinearly viscous liquid in a fundamentally new manner.

The essence of the method consists of taking the non-Newtonian properties of the liquid, determining the viscosity distribution over the cross-sectional area of the channel into account separately from the hydrodynamic characteristics which are common to all liquids.

We write the starting equation (1) as

$$\operatorname{div}(\mu(I_2)\nabla V) = \text{const.} \tag{17}$$

The introduction of $\lambda(\chi_\alpha)$ and the substitution of (6), whose existence and validity have just been proved, reduces Eq. (17) (or (1)) to the equation

$$\operatorname{div}(\lambda(\chi_\alpha)\nabla U) = \text{const.} \tag{18}$$

We also write the equation for the case of the flow of a Newtonian liquid

$$\operatorname{div}(\mu_0\nabla V) = \text{const.} \tag{19}$$

If the region Ω sought is partitioned into infinitely small sections on each of which it may be assumed that $\lambda(\chi_\alpha)$, then it is obvious that Eq. (18) on each of them assumes a Newtonian form (19). Then the significance of the function $U(\chi_\alpha)$ lies in the fact that it describes the velocity distribution for steady-state motion of a liquid with variable viscosity, depending only on the coordinates of a point at the cutoff of the pipe.

In other words, it may be assumed that the flow of a nonlinearly viscous liquid in a cylindrical channel with an arbitrary transverse cross section is divided into the flow of infinitely narrow volumes of immiscible Newtonian liquids with different viscosity.

The choice of $\lambda(\chi_\alpha)$ actually represents the replacement of the internal flow of a nonlinearly viscous liquid by a set of immiscible elementary volumes of Newtonian liquids with a viscosity distribution $\lambda(\chi_\alpha)$ along the cross section of the channel. This flow comprises the "suspension" model of a nonlinearly viscous liquid.

NOTATION

$\chi_\alpha = (\chi_1$ and $\chi_2)$, a point in the two-dimensional ($\alpha = 1,2$) Euclidean space with the coordinates χ_1 and χ_2; z, a coordinate along the channel; $V(\chi_\alpha)$, velocity of the flow in a direction perpendicular to Ω; Ω, a region with the boundary Γ; μ, effective viscosity of the

Newtonian liquid; $\partial P/\partial z$, axial component of the pressure gradient; I_2, second invariant of the strain-rate tensor; F(V), functional to be minimized; $U(\chi_\alpha)$, an auxiliary function which is the solution of the Dirichlet problem for Poisson's equation in the region Ω under study; $\lambda(\chi_\alpha)$, an auxiliary functional depending on the form of the region Ω and the form of the rheological model; $C(\chi_\alpha)$, a constant contour line of $V(\chi_\alpha)$; dS, an element of arc; Γ_1, Γ_2, Γ_3, elements of the contour Γ; Γ_0, one of the lines of the orthogonal coordinate system; P, a point in Ω, lying on one of the contour lines of $U(\chi_\alpha)$; $\tilde{P}$, $\tilde{P}_1$, points lying on Γ_0; $\dot{\gamma}(P, \Gamma_0)$, divergence of the contour lines; P_1, a point lying on the line from the second family of coordinate lines; a and b, half-sides of the rectangle; φ_0, φ_∞, fluidity of the liquid in the limits $\tau \to 0$ and $\tau \to \infty$; τ, intensity of the tangential shear stresses; τ_1, θ, limit and measure of the structural stability of the liquid; n, rheological parameter; ε, fixed error of the iteration method; V, average flow rate of the flow; R, radius of the circle of the semicircular channel; μ_0, Newtonian viscosity; and μ_{-1}, function inverse to μ.

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A FEATURE OF HEAT TRANSFER TO ORGANIC HEAT-TRANSFER MEDIA

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It is shown that the nature of the changes in the wall temperature during heat transfer to an organic heat-transfer medium accompanied by the formation of deposits depends strongly on the roughness of the surface.

Heat transfer to organic liquids at high temperatures of the surfaces cooled by them is accompanied by the formation of carbonaceous deposits [1]. These deposits have very poor thermal conductivities and even in thin layers, they cause a considerable overheating of the walls or heat exchangers and other equipment in which the organic liquids are used as heat-transfer media, coolants, or for other purposes. This problem is particularly important in nuclear reactors. The formation of deposits on the surfaces of fuel elements is one of the main reasons which limits the use of reactors with organic heat-transfer media [2. 3].

Usually the formation of deposits causes a continuous increase of the temperature of the surface being cooled. However, many investigators have observed "anomalous" temperature changes of the surfaces being cooled in some experiments during the formation of deposits on the walls: during the first few minutes when the formation of the deposit occurs most intensively the wall temperature decreases, i.e., heat transfer improves.

This phenomenon has been related to distortions of the experimental results caused by the special features of heating the fuel element by an electric current with a nonuniform formation of the deposit on the wall surface, to the unsteady-state nature of the heat transfer in the first minutes of the experiment, and to various other causes.

We have experimentally established that the nature of the temperature changes of the surface being cooled during the formation of solid deposits on it depends on its roughness. "Anomalous" temperature changes of the wall are observed only in the experiments in which fuel elements with smooth (polished) surfaces are used.

The experiments were carried out in a laboratory heat exchange apparatus, which consisted of a closed hydraulic loop of stainless steel piping. The liquid being investigated was

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